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# ENERGY APPROXIMATIONS OF THE SOLUTION OF THE FIRST PROBLEM OF THE LINEAR THEORY OF ELASTICITY†

V. Ya. TERESHCHENKO

Rostov-on-Don

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The condition for which the Ritz approximations and the Trefftz approximations lead to a single energy approximation of the solution of the first problem of the theory of elasticity is established. © 2000 Elsevier Science Ltd. All rights reserved.

The method of orthogonal projections, which, to solve the Dirichlet problem, is realized by constructing opposed Ritz and Trefftz approximations [1, 2], is the geometrical formulation of the Dirichlet variational principle. This means that the norm of the exact (energy) solution has an upper limit set by the norm of the approximate Ritz solution and has a lower limit set by the norm of the approximate Trefftz solution. These limits are also known as bilateral limits and for the solution $u$ of the Dirichlet problem for the operator $\Delta^c = -\Delta + c$, $c(x) \geqslant \gamma > 0$ ($\Delta$ is the Laplace operator) have the form

$$\|u_n\|_{n=1,2,\ldots} \leqslant \|u_{n+1}\| \leqslant \|u\| \leqslant \|r_{n+1}\| \leqslant \|r_n\|$$

where $\{u_n\}$ are the Trefftz approximations and $\{r_n\}$ are the Ritz approximations, and exists in the form of the norm of the operator $\Delta^c$. it follows from this chain of inequalities that for any number of approximations n in a certain special case the norm of the Ritz approximations is equal to the norm of the Trefftz approximations; consequently, the approximations are identical.

Below we establish the condition for the approximations of the solution of the first problem of the theory of elasticity to be identical, based on a generalization of the Dirichlet variational principle in the linear theory of elasticity.

**1.** The Dirichlet variational principle, from which the method of solving the Dirichlet harmonic problem follows, is generalized in the following formulation (the proof is similar): of all the fairly smooth vector functions of the displacements $\mathbf{u}(x)$, $x \in \bar{G} = G + S$ with a finite (permissible) energy integral, which take a specified value on the boundary of the region, the function for which the energy integral has a minimum value is the homogeneous solution of the equation of the linear theory of elasticity. Correspondingly, from this there follows an energy method of solving the first problem of the theory of elasticity in a bounded region $G$ with a fairly smooth a boundary $S$

$$\mathbf{Au}_0(x) = 0, \quad x \in G; \quad \mathbf{u}_0(x) = \mathbf{u}^*, \quad x \in S \tag{1.1}$$

which is described by the relation

$$I(\mathbf{u}_0) \leqslant I(\mathbf{u}) = 2\int_G W(\mathbf{u})dG \tag{1.2}$$

for all permissible vector functions $\mathbf{u}$ ($u\,|_S = \mathbf{u}^*$) and

$$I(\mathbf{u}_0, \mathbf{v}) = 2\int_G W(\mathbf{u}_0, \mathbf{v})dG = 0 \tag{1.3}$$

for all permissible vector-functions $\mathbf{v}$ ($\mathbf{v}\,|_S = 0$).

Here $2W(\mathbf{u})$ is the generally accepted notation [1] of the quadratic form of the components of the elastic strain tensor $\boldsymbol{\varepsilon}_{ik}(\mathbf{u})$ ($i, k = 1, 2, 3$), $2W[\mathbf{u}, \mathbf{v}]$ is the corresponding bilinear form.

From Eq. (1.3), by virtue of Betti's formula and the density of the set of vector functions $\mathbf{v}$ in vector-function space with integrable square $L_2(G)$ [1] it follows that $A\mathbf{u}_0(x) = 0, x \in G$, simultaneously it follows

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from (1.3) that the sets of homogeneous solutions of the differential equation of the theory of elasticity and sets of vector functions, equal to zero at points on the boundary of the region, are orthogonal in the sense of the "scalar product", defined by the integral $I(\mathbf{u},\mathbf{v})$.

**2.** To prove relations (1.2), the permissible vector functions $\mathbf{u}(x)$ are represented at the points $x \in \overline{G}$ as $\mathbf{u} = \mathbf{u}_0 + t\mathbf{v}$, where $\mathbf{v}$ is an arbitrary real number. Then, the homogeneous solution $\mathbf{u}_0$ can be regarded as the difference of the function $\mathbf{u}$, which takes the same value as the function $\mathbf{u}_0$ on the boundary, and the function $\mathbf{v}$, equal to zero on the boundary $S$ of the region $G$. In this way we form the difference (the summation over i is carried out from $i = 1$ to $i = n$)

$$\mathbf{u}_n = \mathbf{u} - \mathbf{v}_n, \quad \mathbf{v}_n = \sum c_i \boldsymbol{\varphi}_i \left( \mathbf{u}|_S = \mathbf{u}^* \right) \tag{2.1}$$

where $\varphi_i(x), x \in G$ are coordinate vector functions, equal to zero at the points in $S$.

We will determine for what values of the coefficients $c_i$ Eq.(1.3) is satisfied in the form

$$I(\mathbf{u}_n, \mathbf{v}_n) = 2\int_G W(\mathbf{u}_n, \mathbf{v}_n) dG = 0 \tag{2.2}$$

Using the first equation of (2.1) we obtain from (2.2)

$$I(\mathbf{u}, \mathbf{v}_n) - I(\mathbf{v}_n) = 0$$

Using the second equation of (2.1) we therefore have

$$\sum c_i I(\mathbf{u}, \boldsymbol{\varphi}_i) - \sum c_i^2 I(\boldsymbol{\varphi}_i) = 0 \tag{2.3}$$

and, taking into account the fact that $c_i \neq 0$, we obtain

$$c_i = I(\mathbf{u}, \boldsymbol{\varphi}_i)\left[I(\boldsymbol{\varphi}_i)\right]^{-1}, \quad i = 1, \ldots, n \tag{2.4}$$

If the functions $\varphi_2$ are such that

$$I(\varphi_i) = 1 \tag{2.5}$$

it follows from (2.4) that the value of the coefficients *ci* correspond to the projections (in the sense of the " scalar product") of the permissible vector function $\mathbf{u}$, defined by the integral $I$ ($\mathbf{u}$,$\mathbf{v}$), onto the set of vector function $\{\varphi_i\}$, equal to zero on the boundary $S$. Condition (2.5) is satisfied for the vector functions

$$\overline{\boldsymbol{\varphi}}_i = \boldsymbol{\varphi}_i \left( |\boldsymbol{\varphi}_i|_H \right)^{-1}, \quad |\boldsymbol{\varphi}_i|_H = \{I(\boldsymbol{\varphi}_i)\}^{1/2}$$

where $|\cdot|_H$ is the energy norm of the first problem of the theory of elasticity, equivalent to the norm in Sobolev space $\mathring{W}_2^1(G)$ of the vector functions, continuous in $G$, the generalized derivatives of which belong to $L_2(G)$.

We recall that the functions from $\mathring{W}_2^1(G)$ vanish in the generalized sense on the boundary $S$ if the boundary of the region is sufficiently smooth; the functions from $\mathring{W}_2^1(G)$ then vanish on the boundary in the usual sense [1]. The space $\mathring{W}_2^1(G)$ is separable [1], and consequently, a basis system of vector functions $\{\overline{\varphi}_n\}$ exists.

Hence, from (2.4) we obtain the values of the coefficients

$$c_i = I(\mathbf{u}, \overline{\boldsymbol{\varphi}}_i), \quad i = 1, \ldots, n \tag{2.6}$$

and we have relations, which differ from (2.1) by having $\mathbf{v}_n$, $\varphi_i$ replaced by $\overline{\mathbf{v}}_n$, $\overline{\varphi}$ and we finally obtain the sequence $\{\mathbf{u}_n\}$ of solutions of the equation

$$A\mathbf{u}_n(x) = 0, \quad x \in G.$$

According to the scheme of the method of orthogonal projections of the solution of the Dirichlet problem for second-order elliptic equations [1–3] (or the first problem of the theory of elasticity [4]) the above sequence $\{\mathbf{u}_n\}$ corresponds to the Ritz approximations for the solution of problem (1.1). Then the Trefftz approximations correspond [2] to the projection, in the sense of the "scalar product", defined by the integral $I(\mathbf{u},\mathbf{v})$, of the permissible vector function $\mathbf{u}$ ($\mathbf{u}\,|_S = \mathbf{u}$ onto the set $\{\mathbf{u}_n\}$ of the differential equation of problem (1.1). Corresponding to this we form the seminormalized sequence of homogeneous solutions

$$\bar{\mathbf{u}}_n = \mathbf{u}_n\left(I(\mathbf{u}_n)\right)^{-1/2}, \quad \forall n \tag{2.7}$$

(here $I(\mathbf{u}_n) < \infty$ (see below)); the terminology employed is due to the fact that the integral $I(\mathbf{u})$ (see (1.2)) defines a seminorm and not a norm, since it follows from $2W(\mathbf{u}) = 0 \Leftrightarrow \varepsilon_{ik}(\mathbf{u}) = 0$ that $\mathbf{u}$ = const. The approximate Trefftz solution is then (the summation over $n$ is carried from $n = 1$ to $n = N$)

$$\mathbf{u}_N^T = \sum d_n \bar{\mathbf{u}}_n, \quad d_n = I(\mathbf{u}, \bar{\mathbf{u}}_n) \tag{2.8}$$

We will show that this solution is identical to the approximate Ritz solution

$$\mathbf{u}_N^R = \sum \mathbf{u}_n$$

To do this we need to make the following transformations of the integrals

$$I(\mathbf{u}_n) = I(\mathbf{u} - \bar{\mathbf{v}}_n) = I(\mathbf{u}) - 2\sum c_i I(\mathbf{u}, \bar{\varphi}_i) + \sum c_i^2 I(\bar{\varphi}_i) = I(\mathbf{u}) - \sum c_i^2$$

(hence $I(\mathbf{u}_n) < \infty$, since $I(\mathbf{u}) < \infty$ (see (1.2)))

$$I(\mathbf{u}, \mathbf{u}_n) = I(\mathbf{u}, \mathbf{u} - \bar{\mathbf{v}}_n) = I(\mathbf{u}) - I(\mathbf{u}, \bar{\mathbf{v}}_n) =$$
$$= I(\mathbf{u}) - \sum c_i I(\mathbf{u}, \bar{\varphi}_i) = I(\mathbf{u}) - \sum c_i^2$$

We have used relations (2.5) and (2.6) here. Note that a generalization of Bessel's inequality, well-known in the theory of Fourier series [1], follows from the value of the first integral.

As a result, for the approximation solution of (2.8), taking inequality (2.7) into account, and by virtue of equality of the values of the integrals derived above, we have the required result

$$\mathbf{u}_N^T = \sum I(\mathbf{u}, \bar{\mathbf{u}}_n)\bar{\mathbf{u}}_n = \sum I(\mathbf{u}, \mathbf{u}_n)[I(\mathbf{u}_n)]^{-1}\mathbf{u}_n = \mathbf{u}_N^R$$

Hence, if when constructing the Trefftz approximations, we use a system of homogeneous solutions, seminormalized with respect to the metric of the energy integral, these approximations are identical with the Ritz approximations.

For the numerical realization of variational methods of solving boundary-value problems of mathematical physics, a posteriori estimates of the error of the Ritz and Trefftz approximations are used, which follow from the bilateral estimates of the norm of the energy solution [1].

In connection with the above investigations for bilateral estimates of the solution of the first problem of the theory of elasticity, we note that the result obtained showing that the Ritz and Trefftz approximations are identical is a particular result and does not contradict the a posteriori estimate of the Ritz approximations, if we consider as the Trefftz functional the energy integral (which will correspond to the Dirichlet integral in the classical Trefftz method [1]), when the equality sign occurs in the a posteriori estimate (as in the bilateral estimates). In the remaining cases when the generalized Trefftz functionals are used, which may be constructed in a non-unique way for each boundary-value problem [1, 5], the estimate will have an inequality sign.

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